Laminar Flow over a flat Plate – The Blasius Solution

Consider the laminar flow of an incompressible Newtonian fluid over a flat plate. Since the flow is laminar, it is reasonable to seek a steady solution to the governing equations, and since the plate is flat, the pressure gradient will be zero. The resulting governing equations are,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

where we have also neglected viscous dissipation.

In order to simplify the set of governing equations, we introduce a stream function

$$u \equiv \frac{\partial \psi}{\partial y} \tag{4}$$

$$v \equiv -\frac{\partial \psi}{\partial x} \tag{5}$$

Subsitituting 4 and 5 into 1,

$$\frac{\partial^2 \psi}{\partial xy} - \frac{\partial^2 \psi}{\partial xy} \equiv 0 \tag{6}$$

we see that conservation of mass is identically satisfied if we use a stream function to define our velocity components. The stream function is a clever device which implicity enforces mass conservation (continuity). It is also a very convenient way to visualize fluid flows. For these reasons, the stream function is frequently employed in two dimensional fluid mechanics problems.

Next, we want to simplify the momentum equation, and we do this by postulating that there exists some scaling parameter which will reduce all of the local velocity profiles to a single curve. If we can find such a parameter, our two dimensional problem will become a one dimensional problem and will be much more ammenable to analytic solution. Such a scaling, or similarity parameter has been found, and is given by

$$\eta = y\sqrt{\frac{U_\infty}{\nu x}} \tag{7}$$

Now, define a non-dimensional stream function using our similarity parameter,

$$f(\eta) = \frac{\psi}{U_\infty\sqrt{\nu x/U_\infty}} \tag{8}$$

Next, we need to cast our momentum equation, Equation 2 in terms of our similarity parameter, η. We shall convert each term in sequence, and then put them together.

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} \tag{9}$$

From Equation 8,

$$\psi = U_\infty\sqrt{\frac{\nu x}{U_\infty}}f(\eta) \tag{10}$$

and from the definition to η,

$$\frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{\nu x}} \tag{11}$$

Accordingly, Equation 9 simplifies to

$$u = U_\infty\frac{df}{d\eta} \tag{12}$$

Next, consider the v velocity.

$$v = -\frac{\partial \psi}{\partial x} \tag{13}$$

This is a little involved to evaluate, since

$$\psi = U_\infty\sqrt{\frac{\nu x}{U_\infty}}f(\eta) \tag{14}$$

so,

$$v = -\left[U_\infty\sqrt{\frac{\nu x}{U_\infty}}\frac{\partial f}{\partial x} + \frac{U_\infty}{2}\sqrt{\frac{\nu}{U_\infty x}}f\right] \tag{15}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} \tag{16}$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{2}y\sqrt{\frac{U_\infty}{\nu}}x^{-3/2} \tag{17}$$

$$= -\frac{1}{2x}\eta \tag{18}$$

Simplifying,

$$v = \frac{1}{2}\sqrt{\frac{\nu U_\infty}{x}}\left(\eta\frac{\partial f}{\partial \eta} - f\right) \tag{19}$$

Similarly, for $\partial u/\partial x$,

$$\frac{\partial u}{\partial x} = U_\infty \frac{\partial^2 f}{\partial \eta^2}\frac{\partial \eta}{\partial x} \tag{20}$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2x}U_\infty \frac{\partial^2 f}{\partial \eta^2} \tag{21}$$

and for $\partial u/\partial y$

$$\frac{\partial u}{\partial y} = U_\infty \frac{\partial^2 f}{\partial \eta^2}\frac{\partial \eta}{\partial y} \tag{22}$$

$$\frac{\partial u}{\partial y} = U_\infty \sqrt{\frac{U_\infty}{\nu x}}\frac{\partial^2 f}{\partial \eta^2} \tag{23}$$

And finally, for $\partial^2 u/\partial y^2$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_\infty^2}{\nu x}\frac{\partial^3 f}{\partial \eta^3} \tag{24}$$

Substituting each of these terms into the boundary layer momentum equation, Equation 2, we find a third order ordinary differential equation in f,

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0 \tag{25}$$

Or, in simplified notation,

$$2f''' + ff'' = 0 \tag{26}$$

where
f is out non-dimensional stream function,

$$f' = \frac{u}{U_\infty}$$

and

$$f'' = \frac{\partial u}{\partial y}\sqrt{\frac{\nu x}{U_\infty}}\frac{1}{U_\infty} \qquad \text{related to shear stress} \tag{27}$$

Boundary Conditions

In order to solve this equation, we need to impose boundary conditions on it. It is a third order differential equation, and therefor we need three boundary conditions. This is easy to do in our original variables,

$$u(x,0) = 0 \tag{28}$$
$$v(x,0) = 0 \tag{29}$$
$$u(x,\infty) = U_\infty \tag{30}$$

The first condition corresponds easily through Equation to

$$u(x,0) = 0 \longrightarrow f'(0) = \left.\frac{df}{d\eta}\right|_{\eta=0} = 0$$

The second tells us that there is mass flow through the plate, and so this corresponds to the stream function being 0 (or another constant for that matter).

$$v(x,0) = 0 \longrightarrow f(0) = 0$$

The final boundary condition is a little more problematic

$$u(x,\infty) = U_\infty \longrightarrow f'(\infty) = \left.\frac{df}{d\eta}\right|_{\eta=\infty} = 1$$

Let's explore this boundary condition, and determine how to satisfy it. Download the file `blas0.m` from the website. This is a `matlab` function that implements the above differential equation in a form that `matlab` can use with its ODE solvers.

Let's try solving the equation with various boundary conditions and find the case that works for our physics. Not only do we have the problem that the location corresponding to $\eta = \infty$ is unclear, but the ODE solvers integrate from initial conditions only. We have two initial conditions, (f(0) = 0), f;(0) = 0) but we need to convert a third boundary condition to an initial condition for f"(0). We will do this by trial and error.

```
etai = 0;
etaf = 6.8;
[eta1,y1] = ode45('blas',[etai etaf],[0 0 0.1]);
```

look at the data returned by the solver

```
whos eta1
whos y1
```

Note that the function has returned a vector for η and a matrix with three columns for `y1`. The first column is f, the second column is f' and the third column is f". Let's plot these

```
figure
plot(eta1,y1(:,1),'r-')
hold on
plot(eta1,y1(:,2),'g-')
plot(eta1,y1(:,3),'b-')
xlabel('\eta')
ylabel('f, f`, f`` ')
legend('f','f`','f``')
```

Note that f' is our non-dimensional velocity profile, and should go from 0 to 1. In the figure you generated, f' does not come close 1. Repeat the process trying using f"(0) = 0.2 and check again.

```
etai = 0;
etaf = 6.8;
[eta2,y2] = ode45('blas',[etai etaf],[0 0 0.2]);
figure
plot(eta2,y2(:,1),'r-')
hold on
plot(eta2,y2(:,2),'g-')
plot(eta2,y2(:,3),'b-')
xlabel('\eta')
ylabel('f, f`, f`` ')
legend('f','f`','f``')
```

This is better, but still not quite there. Lets increase f"(0) again.

```
etai = 0;
etaf = 6.8;
[eta3,y3] = ode45('blas',[etai etaf],[0 0 0.332]);
figure
plot(eta3,y3(:,1),'r-')
hold on
plot(eta3,y3(:,2),'g-')
plot(eta3,y3(:,3),'b-')
xlabel('\eta')
ylabel('f, f`, f`` ')
legend('f','f`','f``')
```

This appears to be the correct value – the velocity profile starts at 0 and increases to 1. It reaches 1 very close to $\eta = 5.0$. This is the accepted value for the edge of the boundary layer.

We can now use our definition of η to determine the edge of the boundary layer setting y=δ, the boundary layer thickness.

$$5.0 = \delta\sqrt{\frac{U_\infty}{\nu x}} \tag{31}$$

or,

$$\delta = \frac{5.0x}{\sqrt{Re_x}}$$

Where

$$Re_x = \frac{U_\infty x}{\nu}$$

Now, from our single solution, we can determine the velocity profile anywhere in a laminar boundary layer simply by scaling this solution.

$$y = \frac{\eta}{\sqrt{\frac{U_\infty}{\nu x}}}$$

and the velocity is scaled from its definition.

$$u = f' U_\infty$$

Let us consider an airflow with $U_\infty = 3.0m/s, \nu = 1.5 \times 10^{-5} m^2/s$. Let us plot the velocity profile at three different locations on the plate, x=0.25, 2 and 3 m.

We will create a `matlab` function to convert the Blasius solution to dimensional form. Save the following in a text file called `blasvel.m`. Note that the comments are helpful, but not absolutely necessary.

```
function [u, v, y] = blasvel(f,eta,x,uinf,nu)
%function [u, v, y] = blasvel(f,eta,x,uinf,nu)
%
% J.G. Pharoah, Queen's University, February 2003
%
% This function returns the u and v velocity components based on the Blasius solution of
% laminar flow over a flat plate with zero pressure gradient
%
% f,eta are returned by the function blasius
% x     is the desired x location on the plate
% uinf  is the free stream velocity
% nu    is the kinematic viscosity

u = f(:,2)*uinf;                                     % Equation 7.12 Incropera and DeWitt
v = 1/2 *sqrt(nu*uinf/x)*(eta.*f(:,2) - f(:,1));    % Equation 7.13 Incropera and DeWitt
y = eta/sqrt(uinf/(nu*x));
```

We can now use this function with our solution above. To look at the velocity profile at x=1m type

```
[u,v,y] = blasvel(y3,eta3,1.0,3.0,1.5e-5);
figure
plot(u,y)
xlabel('u [m/s]')
ylabel('y [m]')
```

Now lets repeat this for the three locations and place them on a single plot in a nice way.

```
nu = 1.5e-5;uinf=3.0
uscale = 6;
x = 0.25;
[u,v,y] = blasvel(f,eta,x,uinf,nu); % Compute the velocity profiles at x=0.25

figure;
plot(x+u/uscale,y,'r.-');hold on
plot([x x],[0 max(y)],'r')

x=1.0;
[u,v,y] = blasvel(y3,eta3,x,uinf,nu); % Compute the velocity profiles at x=1.0
plot(x+u/uscale,y,'g.-');hold on
plot([x x],[0 max(y)],'g')


x=2.0;
[u,v,y] = blasvel(y3,eta3,x,uinf,nu); % Compute the velocity profiles at x=2.0
plot(x+u/uscale,y,'m.-');hold on
plot([x x],[0 max(y)],'m')


xtemp = 0:.1:2.5; Rex = uinf*xtemp/nu;
delta = 5*xtemp./sqrt(Rex);
plot(xtemp,delta)

xlim([0 2.5]);

plot([0.25 0.25+1/uscale],[.02 .02],'k')
text(0.25+(1/uscale)/2,0.021,'1 m/s','HorizontalAlignment','Center')
ylabel('y [m]');xlabel('x [m]');title('Developing Laminar Boundary Layer')
text(1.25,0.024,'U_{\infty} = 3.0 m/s, \nu = 1.5 x 10^{-5} m^2/s','HorizontalAlignment',
```

We can also investigate the shear stress on the plate, which will tell us how much drag there is on the plate. The shear stress is given by Newton's law of viscosity (for a Newtonian fluid).

$$\tau_{wall} = \mu \left. \frac{du}{dy} \right|_{y=0}.$$

Notice the strong parallel between momentum transport (through shear forces), and heat transfer. The shear stress is related to the velocity gradient, which is very clearly related to f" in our Blasius solution. Since we need to evaluate the velocity gradient at the surface of the plate, it is f"(0) = 0.332 that will give is related to shear stress.

The skin friction coefficient is the non-dimensional wall shear (note the different definition compared to friction factors),

$$C_f = \frac{\tau_{wall}}{1/2\rho U_\infty^2}$$

C_f can calculated by manipulating the non-dimensional variables in the Blasius solution to yield,

$$\tau_{wall} = 0.332 U_\infty \sqrt{\rho \mu U_\infty / x}$$

or

$$C_f(x) = 0.664 Re_x^{-1/2}$$

Now, we can use this data to solve the energy equation and solve convective heat transfer over a flat plate. At some point, however the flow will become turbulent, and these relations will no longer hold.